Herbert–Stevens Problem in the Monocentric City Model

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This note summarizes the Herbert–Stevens formulation of the monocentric city model as a social optimization problem. The main point is that the competitive Alonso–Muth–Mills equilibrium can be recovered from the dual variables of a cost-minimization problem. In the problem, land rent is not a real resource cost paid by society to itself but it is the shadow price of scarce land. The outside rent r_{A}, however, is a real opportunity cost because urban land could otherwise be used in agriculture.

1. Environment

The city is monocentric: all employment is located at the central business district (CBD), indexed by $x = 0$. Distance from the CBD is denoted by $x \geq 0$. Households are homogeneous and the city is closed with fixed total mass $N > 0$. Landowners are absentee, so land-rent income is not rebated to residents. Each household consumes a composite good $z(x) \geq 0$ and residential land $a(x) > 0$. The utility function is $u(z, a)$.

Assumption 1.1. The function $u : \mathbb{R}_+ \times \mathbb{R}_{++} \to \mathbb{R}$ is continuously differentiable, strictly increasing, and strictly quasi-concave. For each $\bar{u}$ and $a > 0$, the minimum composite-good consumption required to attain $\bar{u}$,

$$z(\bar{u}, a) \equiv \min_{z' \geq 0} \quad z' \quad \text{s.t.} \quad u(z', a) \geq \bar{u}, \tag{1}$$

is finite and differentiable on the relevant domain.

The commuting cost is $T(x)$, with $T'(x) > 0$. The benchmark special case is

$$T(x) = tx, \qquad t > 0. \tag{2}$$

The outside land rent is $r_{\mathrm{A}} > 0$. The residential land rent is denoted by $R(x)$. In the social problem below, $R(x)$ appears as the dual value for the land supply constraint.

Let $s(x)$ be the amount of land in the annulus or interval at distance x per unit $\mathrm{d}x$. For example,

$$s(x) = \begin{cases} 1 & \text{linear city,} \\ 2\pi x & \text{circular city.} \end{cases} \tag{3}$$

Household density per unit land is $n(x) \geq 0$. Thus the mass of households between x and $x + \mathrm{d}x$ is $s(x)n(x)\mathrm{d}x$. The local land-capacity constraint is

$$n(x)a(x) \leq 1. \tag{4}$$

2. The Herbert–Stevens Cost-Minimization Problem

Fix a target utility level $\bar{u}$. The Herbert–Stevens problem chooses the density, lot size, composite consumption, and urban boundary to minimize real resource cost. Assume that a city occupies locations satisfying $0 \le x \le \bar{x}$. Then, the problem is

$$\min_{\bar{x},n,z,a} \quad C(n,z,a,\bar{x}) \equiv \int_0^{\bar{x}} s(x)\left(n(x)\left(z(x)+T(x)\right)+r_{\mathrm{A}}\right)\mathrm{d}x \tag{5}$$

$$\text{s.t.} \quad u(z(x),a(x)) \ge \bar{u} \qquad \text{for } 0 \le x \le \bar{x} \text{ with } n(x) > 0, \tag{6}$$

$$n(x)a(x) \le 1 \qquad \text{for } 0 \le x \le \bar{x}, \tag{7}$$

$$\int_0^{\bar{x}} s(x)n(x)\mathrm{d}x = N, \tag{8}$$

$$n(x) \ge 0, \quad z(x) \ge 0, \quad a(x) > 0. \tag{9}$$

The first term in the objective is aggregate composite-good consumption plus aggregate commuting cost. The second term is the opportunity cost of converting land from its outside use to urban residential use. Market land-rent payments are not included in the objective because they are transfers from residents to absentee landowners, not resources used up by the city economy.

Remark 2.1. The formulation in (5) assumes that all land satisfying $0 \le x \le \bar{x}$ is urban land. Equivalently, one may introduce an urban land-use share variable satisfying $0 \le \ell(x) \le 1$, replace (7) by $n(x)a(x) \le \ell(x)$, and write the land opportunity cost as $\int s(x)r_{\mathrm{A}}\ell(x)\mathrm{d}x$. At an optimum with a connected monocentric city, $\ell(x) = 1$ for $0 \le x \le \bar{x}$ and $\ell(x) = 0$ outside.

2.1. Pointwise first-order conditions

Consider an interior location with $n(x) > 0$. The utility constraint is imposed only on occupied locations. This can be represented by the inequality

$$n(x)(\bar{u} - u(z(x),a(x))) \le 0, \tag{10}$$

which is equivalent to (6) wherever $n(x) > 0$ and is vacuous where $n(x) = 0$. The local land-capacity and population constraints are written as

$$n(x)a(x) - 1 \le 0, \qquad N - \int_0^{\bar{x}} s(x)n(x)\mathrm{d}x = 0. \tag{11}$$

For a minimization problem with pointwise inequality constraints of the form $g(x) \le 0$, nonnegative multipliers are added to the objective. Thus $\lambda(x) \ge 0$ is attached to (10), $R(x) \ge 0$ is attached to $n(x)a(x) - 1 \le 0$, and $G \in \mathbb{R}$ is attached to the population equality. This gives

$$\begin{aligned}\mathcal{L} = C(n,z,a,\bar{x}) &+ \int_0^{\bar{x}} s(x)\lambda(x)n(x)(\bar{u} - u(z(x),a(x)))\mathrm{d}x \\ &+ \int_0^{\bar{x}} s(x)R(x)(n(x)a(x)-1)\mathrm{d}x + G\left(N - \int_0^{\bar{x}} s(x)n(x)\mathrm{d}x\right).\end{aligned} \tag{12}$$

Collecting terms inside the integral yields the compact expression

$$\begin{aligned}\mathcal{L} \equiv \int_0^{\bar{x}} s(x)\Big(&n(x)(z(x)+T(x)) + r_{\mathrm{A}} \\ &+ \lambda(x)n(x)(\bar{u} - u(z(x),a(x))) + R(x)(n(x)a(x)-1) - Gn(x)\Big)\mathrm{d}x + GN,\end{aligned} \tag{13}$$

where $\lambda(x) \ge 0$ is the multiplier for the utility constraint, $R(x) \ge 0$ is the multiplier for residential land capacity, and G is the multiplier for the population constraint written as $N - \int s(x)n(x)\mathrm{d}x = 0$.

The variational first-order conditions with respect to $z(x)$, $a(x)$, and $n(x)$ are

$$s(x)n(x)\left(1-\lambda(x)u_z(z(x),a(x))\right)=0, \tag{14}$$

$$s(x)n(x)\left(R(x)-\lambda(x)u_a(z(x),a(x))\right)=0, \tag{15}$$

$$s(x)\left(z(x)+T(x)+R(x)a(x)+\lambda(x)\left(\bar{u}-u(z(x),a(x))\right)-G\right)=0. \tag{16}$$

At an occupied location with $s(x)>0$ and $n(x)>0$, these conditions are equivalent to the same equations after dividing by $s(x)n(x)$ in (14) and (15) and by $s(x)$ in (16). Complementarity gives

$$0\le\lambda(x)\perp u(z(x),a(x))-\bar{u}\ge 0, \tag{17}$$

$$0\le R(x)\perp 1-n(x)a(x)\ge 0, \tag{18}$$

where the convention $0\le A(x)\perp B(x)\ge 0$ indicates $A(x)\ge 0$, $B(x)\ge 0$, and $A(x)B(x)=0$. The outside land use determines the lower envelope of urban land rent: at the endogenous urban fringe $R(\bar{x})=r_{\mathrm{A}}$, and in the occupied interior $R(x)\ge r_{\mathrm{A}}$. Thus $R(x)-r_{\mathrm{A}}$ is differential rent, while $R(x)$ itself is the full residential land rent.

Proposition 2.1. At every occupied interior location, the Herbert–Stevens optimum satisfies

$$u(z(x),a(x))=\bar{u}, \tag{19}$$

$$n(x)a(x)=1, \tag{20}$$

$$\frac{u_a(z(x),a(x))}{u_z(z(x),a(x))}=R(x), \tag{21}$$

$$z(x)+T(x)+R(x)a(x)=G. \tag{22}$$

Proof. Since u is strictly increasing in z, the utility constraint binds at any location with positive population; otherwise $z(x)$ could be reduced. Then (14) and (15) imply (21). If $n(x)>0$, the residential land-rent multiplier is positive in the occupied city, and (18) implies $n(x)a(x)=1$. In a connected occupied urban interval, land with positive residential rent is not left vacant, so (20) holds except possibly at a zero-measure fringe. Substituting the binding utility constraint into (16) gives (22). □

The economic interpretation is immediate. (21) is the household tangency condition between land and the composite good. (22) is the household budget identity with full income G. Thus, under the sign convention in (13), G is the marginal resource value of adding one household while maintaining the required utility $\bar{u}$. In the competitive equilibrium interpretation, G is identified with exogenous household income y. If instead the population constraint were written as $\int s(x)n(x)\mathrm{d}x-N=0$, the corresponding multiplier would be $-G$.

Note: Shadow prices and household cost minimization

For a household at location x, consider the expenditure minimization problem

$$e(R(x),\bar{u})\equiv\min_{z,a}\quad z+R(x)a\quad \text{s.t.}\quad u(z,a)\ge\bar{u},\quad z\ge 0,\quad a>0. \tag{23}$$

The first-order conditions are

$$1-\lambda u_z(z,a)=0,\qquad R(x)-\lambda u_a(z,a)=0, \tag{24}$$

which coincide with (14) and (15). Thus the Herbert–Stevens planner chooses the same lot size and composite consumption as a household facing land rent $R(x)$ and utility target $\bar{u}$. The remaining condition (22) says $e(R(x), \bar{u}) + T(x) = G$.

3. Recovery of the Competitive Monocentric Equilibrium

In the Alonso–Muth–Mills model, a household with income y at location x solves

$$\max_{z,a} \quad u(z,a) \quad \text{s.t.} \quad z + T(x) + R(x)a \leq y, \quad z \geq 0, \quad a > 0. \tag{25}$$

In a spatial equilibrium with homogeneous households, all occupied locations deliver the same utility $\bar{u}$. Comparing (25) with (21) and (22), the competitive equilibrium conditions are recovered by setting

$$y = G. \tag{26}$$

The dual variable $R(x)$ is the residential land rent supporting the allocation, while G is the common full income needed to support utility $\bar{u}$ at all occupied locations.

Proposition 3.1. Suppose an allocation solves the Herbert–Stevens problem and satisfies the regularity conditions in Assumption 1.1. Set $y = G$ and interpret $R(x)$ as the market land rent. Then each occupied location satisfies household utility maximization, land-market clearing, equal utility, and the population constraint of the closed-city Alonso–Muth–Mills equilibrium.

Proof. The household first-order conditions are (21) and the binding budget equation (22) with $y = G$. Strict quasi-concavity of u gives household optimality. Land-market clearing is (20). Equal utility is (19). The closed-city population condition is exactly (8). □

The reverse direction is also useful. Given a competitive equilibrium with utility $\bar{u}$, income y, rent profile $R(x)$, density $n(x)$, and lot size $a(x)$, the equilibrium conditions imply the Herbert–Stevens first-order conditions with $G = y$. Thus the social optimum and competitive equilibrium are dual descriptions of the same monocentric allocation under the maintained assumptions.

4. Bid Rent

For each utility level $\bar{u}$ and land consumption $a > 0$, let $z(\bar{u}, a)$ be defined by (1). The bid-rent function is

$$\Psi(x, \bar{u}) \equiv \max_{a>0} \quad \frac{y - T(x) - z(\bar{u}, a)}{a}. \tag{27}$$

When it is useful to emphasize the dependence on income, write $\Psi(x, \bar{u}; y)$. The maximizer is denoted by

$$a^*(x, \bar{u}) \in \arg\max_{a>0} \frac{y - T(x) - z(\bar{u}, a)}{a}. \tag{28}$$

Lemma 4.1. If $a^*(x, \bar{u})$ is an interior maximizer of (27), then

$$y - T(x) - z(\bar{u}, a^*) = \Psi(x, \bar{u})a^*, \tag{29}$$

$$z_a(\bar{u}, a^*) + \Psi(x, \bar{u}) = 0. \tag{30}$$

Moreover,

$$\Psi_x(x, \bar{u}) = -\frac{T'(x)}{a^*(x, \bar{u})} < 0. \tag{31}$$

Proof. The first equation is the definition of Ψ evaluated at the maximizing lot size. The first-order condition for a is

$$\frac{-z_a(\bar{u}, a)a - (y - T(x) - z(\bar{u}, a))}{a^2} = 0. \tag{32}$$

Using (29) yields (30). The envelope theorem gives (31). □

The bid-rent curve is decreasing in distance because commuting cost is increasing. For the benchmark $T(x) = tx$, the slope is

$$\Psi_x(x, \bar{u}) = -\frac{t}{a^*(x, \bar{u})}. \tag{33}$$

Proposition 4.1. At an occupied equilibrium location with $n(x) > 0$,

$$R(x) = \Psi(x, \bar{u}; y). \tag{34}$$

Proof. The equilibrium budget identity is

$$y - T(x) - z(x) = R(x)a(x). \tag{35}$$

Since $u(z(x), a(x)) = \bar{u}$, we have $z(x) = z(\bar{u}, a(x))$. Therefore

$$R(x) = \frac{y - T(x) - z(\bar{u}, a(x))}{a(x)} \leq \Psi(x, \bar{u}; y). \tag{36}$$

If the inequality were strict, the household could attain utility $\bar{u}$ with the same income while paying a land rent higher than $R(x)$, contradicting the household tangency condition and strict quasi-concavity. Hence equality holds. □

5. Total Differential Rent

For a fixed utility level $\bar{u}$ and income y, land at distance x is urban if its residential bid rent exceeds the outside rent:

$$\Psi(x, \bar{u}; y) \geq r_{\mathrm{A}}. \tag{37}$$

Because $\Psi_x < 0$, the occupied city is connected and has boundary $\bar{x}$ satisfying

$$\Psi(\bar{x}, \bar{u}; y) = r_{\mathrm{A}}. \tag{38}$$

This is the urban fringe condition. At the fringe, the best residential use of land earns exactly the outside rent. Inside the city, the differential rent $\Psi(x, \bar{u}; y) - r_{\mathrm{A}}$ is positive; outside the city, it is nonpositive and land remains in the outside use.

The total differential rent in a monocentric city is

$$\mathrm{TDR}(\bar{u}) = \int_0^{\bar{x}} s(x)\left(\Psi(x,\bar{u};y) - r_{\mathrm{A}}\right)\mathrm{d}x. \tag{39}$$

Proposition 5.1. For a fixed $(y, \bar{u})$, choosing urban land to maximize total differential rent,

$$\max_{\bar{x}\geq 0} \quad \int_0^{\bar{x}} s(x)\left(\Psi(x,\bar{u};y) - r_{\mathrm{A}}\right)\mathrm{d}x, \tag{40}$$

is equivalent to the spatial land-use part of the Herbert–Stevens problem. The first-order condition for an interior boundary is the fringe condition (38).

Proof. At each location, the maximum willingness to pay for land while maintaining utility $\bar{u}$ is $\Psi(x, \bar{u}; y)$. The opportunity cost of urban land is r_{A}. Thus the net surplus from assigning land at x to urban use is $\Psi(x, \bar{u}; y) - r_{\mathrm{A}}$. Since $\Psi_x < 0$, the maximizing occupied set is the interval on which this surplus is nonnegative. Differentiating (40) with respect to $\bar{x}$ gives

$$s(\bar{x})\left(\Psi(\bar{x},\bar{u};y) - r_{\mathrm{A}}\right) = 0, \tag{41}$$

which gives (38) when $s(\bar{x}) > 0$. □

The closed-city population condition determines the equilibrium utility level. Given $(y, \bar{u})$, the density induced by the bid-rent solution is

$$n(x) = \frac{1}{a^*(x,\bar{u})} \qquad \text{for } 0 \leq x \leq \bar{x}, \tag{42}$$

and the closed-city condition is

$$N = \int_0^{\bar{x}} s(x)\frac{1}{a^*(x,\bar{u})}\mathrm{d}x. \tag{43}$$

Because higher $\bar{u}$ requires larger compensated consumption and lowers bid rent, (43) is the equation that selects $\bar{u}$ for a closed city with fixed N and income y.

6. Total Differential Rent and the Cost Value

Define the minimized Herbert–Stevens cost value by

$$C(\bar{u},N) \equiv \min_{\bar{x},n,z,a} \quad \int_0^{\bar{x}} s(x)\left(n(x)\left(z(x)+T(x)\right)+r_{\mathrm{A}}\right)\mathrm{d}x \tag{44}$$

$$\text{s.t.} \quad u(z(x),a(x)) \geq \bar{u}, \quad n(x)a(x) \leq 1, \quad \int_0^{\bar{x}} s(x)n(x)\mathrm{d}x = N.$$

Here $C(\bar{u}, N)$ is a real resource cost: it includes composite-good consumption, commuting cost, and the agricultural opportunity cost of urban land, but it does not include land-rent payments.

Proposition 6.1. In the closed-city equilibrium with absentee landlords, income y, population N, and common utility $\bar{u}$, total differential rent satisfies

$$\mathrm{TDR}(\bar{u},N) = yN - C(\bar{u},N). \tag{45}$$

Proof. At each occupied location, the household budget condition is

$$z(x) + T(x) + R(x)a(x) = y. \tag{46}$$

Multiplying by the mass of households at distance x, integrating over the city, and using the population constraint gives

$$\begin{aligned} yN &= \int_0^{\bar{x}} s(x)n(x)\left(z(x) + T(x) + R(x)a(x)\right) \mathrm{d}x \\ &= \int_0^{\bar{x}} s(x)n(x)\left(z(x) + T(x)\right) \mathrm{d}x + \int_0^{\bar{x}} s(x)R(x)n(x)a(x)\mathrm{d}x. \end{aligned} \tag{47}$$

Land-market clearing gives $n(x)a(x) = 1$ on occupied urban land. Therefore

$$\begin{aligned} yN &= \int_0^{\bar{x}} s(x)n(x)\left(z(x) + T(x)\right) \mathrm{d}x + \int_0^{\bar{x}} s(x)R(x)\mathrm{d}x \\ &= C(\bar{u}, N) + \int_0^{\bar{x}} s(x)\left(R(x) - r_{\mathrm{A}}\right) \mathrm{d}x. \end{aligned} \tag{48}$$

Since $R(x) = \Psi(x, \bar{u}; y)$ in equilibrium, the last integral is $\mathrm{TDR}(\bar{u}, N)$. This proves (45). □

The identity has a simple interpretation. Aggregate household income yN is divided into real resource cost $C(\bar{u}, N)$ and differential land rent. Because landlords are absentee, differential rent is not part of residents' disposable income. It is nevertheless generated by the same spatial allocation, and it is exactly the residual between aggregate income and the real cost of supporting N households at utility $\bar{u}$.

Remark 6.1. In a closed city, N is fixed and yN is fixed for a given income y. Thus minimizing $C(\bar{u}, N)$ at a target utility is equivalent to maximizing $\mathrm{TDR}(\bar{u}, N)$, because the two objectives differ only by the constant yN.

This identity also clarifies the connection with an open-city model. In an open city, the utility level $\bar{u}$ is fixed by an outside option and population is endogenous. If landlords are absentee and there is no fiscal instrument, households solve the private problem with budget income y and land rents leave the city.

To relate the open-city allocation to the differential-rent formulation, introduce a lump-sum fiscal term Q per household. Use the sign convention that $Q > 0$ is a subsidy paid to households and $Q < 0$ is a tax. The household budget constraint becomes

$$z(x) + T(x) + R(x)a(x) \leq y + Q. \tag{49}$$

The corresponding bid rent is

$$\Psi_Q(x, \bar{u}) \equiv \max_{a>0} \frac{y + Q - T(x) - z(\bar{u}, a)}{a}. \tag{50}$$

If differential rent is collected and rebated equally to residents, the balanced-budget condition is

$$Q = \frac{\mathrm{TDR}(\bar{u}, N)}{N}. \tag{51}$$

In this expression, $\mathrm{TDR}(\bar{u}, N)$ is evaluated at the rent profile induced by the bid rent Ψ_Q and the associated open-city population N. Equivalently, if one writes the fiscal instrument as a tax τ paid by residents, then $\tau = -Q$ under this sign convention.

With the transfer included, the same accounting identity becomes

$$\mathrm{TDR}(\bar{u}, N) = (y + Q)N - C(\bar{u}, N). \tag{52}$$

Combining (51) and (52) gives

$$C(\bar{u}, N) = yN. \tag{53}$$

Thus, in the open-city interpretation with equal redistribution of differential rent, population adjusts until the real resource cost of sustaining the resident population at the externally given utility equals aggregate base income. The fiscal term Q is not a new real resource; it is an accounting device that determines whether differential land rent accrues to absentee landlords or is returned to residents.

7. Payments, Resource Costs, and Rents

It is important to distinguish four objects.

1. *Household budget payments* at location x are

$$z(x) + T(x) + R(x)a(x) = y. \tag{54}$$

   These are the private expenditures faced by a household.
2. *Real resource costs per household* are

$$z(x) + T(x). \tag{55}$$

   These are composite-good consumption and commuting resources.
3. *Land opportunity cost* is r_{A} per unit of land. This is included in the social objective because urban land displaces the outside use.
4. *Differential rent* is

$$R(x) - r_{\mathrm{A}} = \Psi(x, \bar{u}; y) - r_{\mathrm{A}}. \tag{56}$$

   It is the excess of residential land value over the outside rent.

The market land rent $R(x)$ is not itself a real resource cost. It is a price that decentralizes the allocation of scarce land. Only the opportunity cost r_{A} enters the social resource objective directly; the remaining part $R(x) - r_{\mathrm{A}}$ is a scarcity rent generated by the location advantage of land inside the city.